ON THE THEORY OF DIRECT GYROSCOPIC STABILIZERS

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The direct gyroscopic stabilizers used to reduce the rolling of vessels and to stabilize various unstable objects are also applied as stabilizers of platforms on ships and on planes. This paper investigates motions of direct gyroscopic stabilizers of the passive and of the active kind under the condition of irregular rolling of ships and presents an estimate of the accuracy of stabilization when the rolling is a stationary random phenomenon.

1. The passive gyroscopic stabilizer for irregular ship-rolling. The equations of motion of a passive gyroscopic stabilizer for a rolling ship are as follows [1]:

$$A\alpha'' + m\alpha' + H\beta' + lP\alpha = a\theta'' + m\theta' \qquad \left(a = \frac{lP}{g}\, r\right)$$
$$B\beta'' + E\beta' - H\alpha' + \varkappa\beta = 0 \tag{1.1}$$

Here α is the rotation angle of the gyro-stabilizer's outer frame about its axis, β is the rotation angle of the gyro-stabilizer's inner frame (gyrocamera) about its axis, θ is the angle of the ship's roll, A and B are the respective moments of inertia, H is the gyroscope's angular momentum, m and E are coefficients of viscous friction, lP is the static moment of the gyro-stabilizer's outer frame, κ is the rigidity of the spring connecting the gyro-stabilizer's inner frame (gyrocamera) with the outer frame, r is the distance between the center of the ship's roll and the axis of the gyro-stabilizer. We shall introduce the following matrices:

$$f(D) = \left\| \begin{matrix} D^2 + \frac{m}{A} D + \frac{lP}{A} & \frac{H}{A} D \\ -\frac{H}{B} D & D^2 + \frac{E}{B} D + \frac{\varkappa}{B} \end{matrix} \right\| \left(D = \frac{d}{dt}\right) \tag{1.2}$$

$$y=\left\| \begin{matrix} \alpha \\ \beta \end{matrix} \right\|, \qquad e(D)=\left\| \begin{matrix} \frac{a}{A}D^2+\frac{m}{A}D \\ 0 \end{matrix} \right\| \tag{1.3}$$

The system of scalar differential equations (1.1) can be replaced by the matrix equation

$$f(D)\,y=e(D)\theta(t) \tag{1.4}$$

From Equation (1.4) we obtain

$$y=\frac{F(D)\,e(D)}{\Delta(D)}\theta(t)=Y(D)\,\theta(t) \tag{1.5}$$

where the matrix $F(D)$ is the adjoint of the matrix $f(D)$

$$F(D)=\left\| \begin{matrix} D^2+\frac{E}{B}D+\frac{\varkappa}{B} & -\frac{H}{A}D \\ \frac{H}{B}D & D^2+\frac{m}{A}D+\frac{lP}{A} \end{matrix} \right\| \tag{1.6}$$

and $\Delta(D)$ is the determinant of the matrix $f(D)$

$$\Delta D=D^4+(\zeta_1+\zeta_2)D^3+(n_1^2+n_2^2+q^2+\zeta_1\zeta_2)D^2+(\zeta_1 n_2^2+\zeta_2 n_1^2)D+n_1^2n_2^2 \tag{1.7}$$

Here

$$n_1^2=\frac{lP}{A}, \quad n_2^2=\frac{\varkappa}{B}, \quad q^2=\frac{H^2}{AB}, \quad \zeta_1=\frac{m}{A}, \quad \zeta_2=\frac{E}{B} \tag{1.8}$$

The quantities n_1 and n_2 are the frequencies of free vibrations of the gyro-stabilizer's outer frame and of the inner frame (gyrocamera), respectively, when $H=0$, that is, when the gyroscope's rotor does not spin. For sufficiently large values of H the frequency of the nutational vibrations of the system is close to q, and the frequency of the precessional vibrations is close to

$$k=\frac{\sqrt{\varkappa lP}}{H} \tag{1.9}$$

It is seen from the Hurwitz condition that all zeros of the polynomial (1.7) are located in the left halfplane of the complex variable D.

According to (1.5) the matrix transfer function $Y(D)$ has the form

$$Y(D)=\frac{1}{\Delta(D)}\left\| \begin{matrix} (D^2+\zeta_2 D+n_2^2)(a^*D^2+\zeta_1 D) \\ \rho a^* D^3+\rho\zeta_1 D^2 \end{matrix} \right\| \qquad \left(a^*=\frac{a}{A},\ \rho=\frac{H}{B}\right) \tag{1.10}$$

According to (1.5) and (1.10) the angles α and β which are the rotation angles of the outer and of the inner frames of the gyro-stabilizer, respectively, can be determined through the operator expressions

$$\alpha = Y_{11}(D)\,\theta(t) = \frac{1}{\Delta(D)}\,[a^*D^4 + (\zeta_1 + a^*\zeta_2)D^3 + (\zeta_1\zeta_2 + a^*n_2^2)D^2 + \zeta_1 n_2^2 D]\,\theta(t) \quad (1.11)$$

$$\beta = Y_{21}(D)\,\theta(t) = \frac{1}{\Delta(D)}\,(\varrho a^* D^3 + \varrho\zeta_1 D^2)\,\theta(t) \quad (1.12)$$

The variances $\overline{\alpha}^2$ and $\overline{\beta}^2$ of the stabilization angle and of the gyro-camera's rotation angle, respectively, equal

$$\overline{\alpha^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y_{11}(i\omega)|^2 S_1(\omega)\,d\omega, \qquad \overline{\beta^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y_{21}(i\omega)|^2 S_1(\omega)\,d\omega \quad (1.13)$$

where $S_1(\omega)$ is the spectral density of the angle of roll which has the form [2]

$$S_1(\omega) = L_1 \frac{4\mu\nu^2}{(\omega^2 - \nu^2)^2 + 4\mu^2\omega^2} \quad (1.14)$$

Expression (1.13) can be transformed into

$$\overline{\alpha^2} = 4\mu\nu^2 L_1 I_6, \qquad \overline{\beta^2} = 4\mu\nu^2 L_1 J_6 \quad (1.15)$$

where

$$I_6 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{g(i\omega)}{h(i\omega)\,h(-i\omega)}\,d\omega, \qquad J_6 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{G(i\omega)}{h(i\omega)\,h(-i\omega)}\,d\omega \quad (1.16)$$

$$\begin{aligned} g(i\omega) &= b_0(i\omega)^{10} + b_1(i\omega)^8 + b_2(i\omega)^6 + b_3(i\omega)^4 + b_4(i\omega)^2 + b_5 \\ G(i\omega) &= B_0(i\omega)^{10} + B_1(i\omega)^8 + B_2(i\omega)^6 + B_3(i\omega)^4 + B_4(i\omega)^2 + B_5 \\ h(i\omega) &= a_0(i\omega)^6 + a_1(i\omega)^5 + a_2(i\omega)^4 + a_3(i\omega)^3 + a_4(i\omega)^2 + a_5(i\omega) + a_6 \end{aligned} \quad (1.17)$$

and the coefficients in the polynomials (1.17) are

$$\begin{aligned} &b_0 = 0, \quad b_1 = a^{*2}, \quad b_2 = 2a^{*2}n_2^2 - \zeta_1^2 - a^{*2}\zeta_2^2 \\ &b_3 = \zeta_1^2\zeta_2^2 - 2\zeta_1^2 n_2^2 + a^{*2}n_2^4, \quad b_4 = -\zeta_1^2 n_2^4, \quad b_5 = 0 \\ &B_0 = 0, \quad B_1 = 0, \quad B_2 = -\rho^2 a^{*2}, \quad B_3 = \rho^2\zeta_1^2, \quad B_4 = 0, \quad B_5 = 0 \end{aligned} \quad (1.18)$$

$$\begin{aligned} a_0 &= 1, \quad a_1 = 2\mu + \zeta_1 + \zeta_2 \\ a_2 &= \nu^2 + 2\mu(\zeta_1 + \zeta_2) + n_1^2 + n_2^2 + q^2 + \zeta_1\zeta_2 \\ a_3 &= \nu^2(\zeta_1 + \zeta_2) + 2\mu(n_1^2 + n_2^2 + q^2 + \zeta_1\zeta_2) + \zeta_1 n_2^2 + \zeta_2 n_1^2 \\ a_4 &= \nu^2(n_1^2 + n_2^2 + q^2 + \zeta_1\zeta_2) + 2\mu(\zeta_1 n_2^2 + \zeta_2 n_1^2) + n_1^2 n_2^2 \\ a_5 &= \nu^2(\zeta_1 n_2^2 + \zeta_2 n_1^2) + 2\mu n_1^2 n_2^2 \\ a_6 &= \nu^2 n_1^2 n_2^2 \end{aligned} \quad (1.19)$$

In the case when all the zeros of the polynomial $h(D)$ are located on the left halfplane of the complex variable D, then according to Phillips [3] the integrals (1.16) have the following form:

$$I_6 = -\frac{M^*}{2a_0 N}, \qquad J_6 = -\frac{M^{**}}{2a_0 N} \quad (1.20)$$

where

$$N = \begin{vmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\ 0 & a_6 & a_5 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_6 & a_5 & a_4 \\ 0 & 0 & 0 & 0 & 0 & a_6 \end{vmatrix} \tag{1.21}$$

and M^* and M^{**} are determinants obtained by interchanging in the determinant (1.21) the first column by a column whose elements are b_0, b_1, ..., b_5, or B_0, B_1, ..., B_5, respectively.

As an example we shall consider a gyroscopic stabilizer whose parameters have the following values:

$$n_1 = 1 \text{ sec}^{-1}, \quad \zeta_1 = 0.6 \text{ sec}^{-1}, \qquad q = 21.2 \text{ sec}^{-1}$$
$$a^* = 0.306, \qquad \varrho = 750 \text{ sec}^{-1}, \qquad n_2 = 0.8 \text{ sec}^{-1}, \qquad \zeta_2 = 0.4 \text{ sec}^{-1}$$

The parameters determining the spectral density of the angle of ship's rolling are taken to be

$$\mu = 0.1 \text{ sec}^{-1}, \qquad \nu = 0.8 \text{ sec}^{-1}$$

The variance of the angle of roll is $L_1 = 0.09$, the standard [mean quadratic] deviation of the angle of roll is $\theta = \sqrt{L_1} = 0.3$, that is, about $18°$.

With these numerical data the variances of the stabilization angle and of the rotation angle of the gyrocamera are, according to (1.15)

$$\bar{\alpha}^2 = 14.10^{-6}, \qquad \bar{\beta}^2 = 0.13$$

The standard deviations of the stabilization angle α^* and of the rotation angle of the gyrocamera β^* are

$$\alpha^* = 3.73 \cdot 10^{-3} \approx 12.8', \qquad \beta^* = 0.36 \approx 20.5°$$

If the values of the parameters do not differ much from the values used in the above example we can use the following approximate formulas for standard deviations of the stabilization angle and of the gyrocamera's rotation angle:

$$\alpha^* \approx \nu\theta^* \left\{\frac{2\mu AB}{H^2}\left[\left(\frac{rn_1^2}{g}\right)^2 \frac{1}{\zeta_1+\zeta_2} + \frac{\zeta_1^2 n_2^4}{\nu^4(\zeta_1 n_2^2 + \zeta_2 n_1^2)}\right]\right\}^{\frac{1}{2}}$$
$$\beta^* \approx \frac{\nu\theta^* A}{H}\left[\left(\frac{rn_1^2}{g}\right)^2 \frac{2\mu+\zeta_1+\zeta_2}{\zeta_1+\zeta_2} + \frac{\zeta_1^2}{\nu^2} + \frac{2\mu\zeta_1 n_1^2 n_2^2}{\nu^4(\zeta_1 n_2^2+\zeta_2 n_1^2)}\right]^{\frac{1}{2}} \tag{1.22}$$

For our example we obtain from Formulas (1.22)

$$\alpha^* \approx 3.76 \cdot 10^{-3}, \qquad \beta^* \approx 0.382$$

2. Active gyroscopic stabilizer for irregular ship's roll. The equations of motion of an active gyroscopic stabilizer for ship's roll are [4]

$$A\alpha'' + m\alpha' + H\beta' + lP\alpha = a\theta'' + m\theta' \quad \left(a = \frac{lP}{g} r\right)$$
$$B\beta'' - H\alpha' = M_y \qquad (M_y = N\alpha' - E\beta') \tag{2.1}$$

Here α is the rotation angle of the outer frame of the gyro-stabilizer about its axis, β is the rotation angle of the gyrocamera about its axis, θ is the angle of ship's roll, A and B are the respective moments of inertia, H is the angular momentum of the gyroscope, m is the coefficient of viscous friction, lP is the static moment of the gyro-stabilizer's outer frame, r is the distance between the center of roll and the axis of the gyro-stabilizer's outer frame, M_y is the moment about the axis of the gyrocamera exerted by an auxiliary electric motor. This motor is controlled by a gyroscopic tachometer measuring angular velocities of the gyro-stabilizer's outer frame.

Introducing matrices

$$f(D) = \left\| \begin{matrix} D^2 + \dfrac{m}{A} D + \dfrac{lP}{A} & \dfrac{H}{A} D \\ -\dfrac{H+N}{B} D & D^2 + \dfrac{E}{B} D \end{matrix} \right\| \tag{2.2}$$

$$y = \left\| \begin{matrix} \alpha \\ \beta \end{matrix} \right\|, \qquad e(D) = \left\| \begin{matrix} \dfrac{a}{A} D^2 + \dfrac{m}{A} D \\ 0 \end{matrix} \right\| \tag{2.3}$$

where $D = d/dt$, we replace the scalar system of differential equations (2.1) by the matrix differential equation

$$f(D)\, y = e(D)\, \theta(t) \tag{2.4}$$

From Equation (2.4) we obtain

$$y = \frac{F(D)\, e(D)}{\Delta(D)} \theta(t) = Y(D)\, \theta(t) \tag{2.5}$$

where $F(D)$ is the adjoint matrix of $f(D)$

$$F(D) = \left\| \begin{matrix} D^2 + \dfrac{E}{B} D & -\dfrac{H}{A} D \\ \dfrac{H+N}{B} D & D^2 + \dfrac{m}{A} D + \dfrac{lP}{A} \end{matrix} \right\| \tag{2.6}$$

and $\Delta(D)$ is the determinant of the matrix $f(D)$

$$\Delta(D) = D\,\Delta_1(D) \tag{2.7}$$
$$\Delta_1(D) = D^3 + (\zeta_1 + \zeta_2) D^2 + (\zeta_1\zeta_2 + n^2 + q^2) D + \zeta_2 n^2$$

Here

$$n^2 = \frac{lP}{A}, \qquad q^2 = \frac{H(H+N)}{AB}, \qquad \zeta_1 = \frac{m}{A}, \qquad \zeta_2 = \frac{E}{B} \tag{2.8}$$

It follows from the Hurwitz condition that all the zeros of the $\Delta_1(D)$ polynomial are located on the left halfplane of the complex variable D.

According to (2.5) the matrix transfer function $Y(D)$ of the system has the form

$$Y(D) = \frac{1}{\Delta_1(D)} \left\| \begin{matrix} a^*D^3 + (\zeta_1 + \zeta_2 a^*) D^2 + \zeta_1\zeta_2 D \\ \rho a^* D^2 + \rho\zeta_1 D \end{matrix} \right\| \qquad \left(a^* = \frac{a}{A}, \quad \rho = \frac{H+N}{B}\right) \tag{2.9}$$

According to (2.5) and (2.9) the rotation angles of the gyrostabilizer α and of the gyrocamera β are

$$\alpha = Y_{11}(D)\theta(t) = \frac{1}{\Delta_1(D)} [a^*D^3 + (\zeta_1 + \zeta_2 a^*) D^2 + \zeta_1\zeta_2 D]\, \theta(t) \tag{2.10}$$

$$\beta = Y_{21}(D)\theta(t) = \frac{1}{\Delta_1(D)} \rho(a^*D^2 + \rho\zeta_1 D)\, \theta(t) \tag{2.11}$$

Variances of the stabilization angle $\overline{\alpha}^2$ and of the gyrocamera's rotation angle β^2 are

$$\overline{\alpha^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y_{11}(i\omega)|^2 S_1(\omega)\, d\omega, \qquad \overline{\beta^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y_{21}(i\omega|^2 S_1(\omega)\, d\omega \tag{2.12}$$

where $S_1(\omega)$ is the spectral density of the angle of roll θ, which, according to (1.14), has the form

$$S_1(\omega) = L_1 \frac{4\mu\nu^2}{(\omega^2 - \nu^2)^2 + 4\mu^2\omega^2}$$

Expression (2.12) can be transformed into

$$\overline{\alpha}^2 = 4\mu\nu^2 L_1 I_5, \qquad \overline{\beta}^2 = 4\mu\nu^2 L_1 J_5 \tag{2.13}$$

where

$$I_5 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{g(i\omega)}{h(i\omega)\, h(-i\omega)}\, d\omega, \qquad J_5 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{G(i\omega)}{h(i\omega)\, h(-i\omega)}\, d\omega \tag{2.14}$$

$$\begin{aligned} g(i\omega) &= b_0 (i\omega)^8 + b_1 (i\omega)^6 + b_2 (i\omega)^4 + b_3 (i\omega)^2 + b_4 \\ G(i\omega) &= B_0 (i\omega)^8 + B_1 (i\omega)^6 + B_2 (i\omega)^4 + B_3 (i\omega)^2 + B_4 \\ h(i\omega) &= a_0 (i\omega)^5 + a_1 (i\omega)^4 + a_2 (i\omega)^3 + a_3 (i\omega)^2 + a_4 (i\omega) + a_5 \end{aligned} \tag{2.15}$$

and the coefficients of the polynomials (2.15) are

$$\begin{gathered} b_0 = 0, \quad b_1 = -a^{*2}, \quad b_2 = \zeta_1^2 + \zeta_2^2 a^{*2}, \quad b_3 = -\zeta_1^2\zeta_2^{\ 2}, \quad b_4 = 0 \\ B_0 = 0, \quad B_1 = 0, \quad B_2 = \rho^2 a^{*2}, \quad B_3 = -\rho^2\zeta_1^2, \quad B_4 = 0 \end{gathered} \tag{2.16}$$

$$\begin{aligned} &a_0 = 1, \quad a_1 = 2\mu + \zeta_1 + \zeta_2 \\ &a_2 = 2\mu(\zeta_1 + \zeta_2) + \zeta_1\zeta_2 + \nu^2 + n^2 + q^2 \\ &a_3 = 2\mu(\zeta_1\zeta_2 + n^2 + q^2) + \nu^2(\zeta_1 + \zeta_2) + \zeta_2 n^2 \\ &a_4 = \nu^2(\zeta_1\zeta_2 + n^2 + q^2) + 2\mu\zeta_2 n^2, \quad a_5 = \zeta_2 n^2 \nu^2 \end{aligned} \tag{2.17}$$

According to Phillips [3] the integrals (2.14) have the following form:

$$I_5 = \frac{M_5^*}{2a_0\Delta_5}, \qquad J_5 = \frac{M_5^{**}}{2a_0\Delta_5} \tag{2.18}$$

where

$$\begin{aligned} M_5^* &= a_0 b_1(a_3a_4 - a_2a_5) + a_0b_2(a_0a_5 - a_1a_4) + a_0b_3(a_1a_2 - a_0a_3) \\ M_5^{**} &= a_0B_2(a_0a_5 - a_1a_4) + a_0B_3(a_1a_2 - a_0a_3) \\ \Delta_5 &= a_0^2a_5^2 - 2a_0a_1a_4a_5 - a_0a_2a_3a_5 + a_0a_3^2a_4 + a_1^2a_4^2 + a_1a_2^2a_5 - a_1a_2a_3a_4 \end{aligned} \tag{2.19}$$

According to (2.13) and (2.18) standard deviations of the stabilization angle of the gyrocamera's rotation angle are

$$\alpha^* = \sqrt{\frac{4\mu\nu^2L_1M_5^*}{2a_0\Delta_5}}, \qquad \beta^* = \sqrt{\frac{4\mu\nu^2L_1M_5^{**}}{2a_0\Delta_5}} \tag{2.20}$$

As an example we shall examine a gyroscopic stabilizer, whose parameters have the following values:

$$\begin{gathered} A = 50\,\text{kgm sec}^2 \quad B = 0.04\,\text{kgm sec}^2 \quad H = 30\,\text{kgm sec} \\ lP = 50\,\text{kgm}, \quad r = 3\,\text{m}, \quad N = 60\,\text{kgm sec}, \quad \zeta_1 = \zeta_2 = 0.6\,\text{sec}^{-1} \end{gathered}$$

Further, according to (2.2), (2.8), and (2.10)

$$\begin{gathered} n = 1\,\text{sec}^{-1}, \quad q^2 = 1350\,\text{sec}^{-2}, \quad a = 15.3\,\text{kgm sec}^2 \\ a^* = 0.306, \quad \rho = 2250\,\text{sec}^{-1} \end{gathered}$$

The parameters determining the spectral density of the roll will be taken as

$$\mu = 0.1\,\text{sec}^{-1} \qquad \nu = 0.8\,\text{sec}^{-1}$$

The variance of the angle of roll is $L_1 = 0.09$, the standard deviation of the angle of roll is $\theta^* = \sqrt{L_1} = 0.3$, that is, approximately 18°.

For these numerical data the variances of the stabilization angle $\overline{\alpha}^2$ and of the gyrocamera's rotation angle $\overline{\beta}^2$ are, according to (2.13)

$$\overline{\alpha^2} = 0.66\cdot 10^{-6}, \qquad \overline{\beta^2} = 0.106$$

The standard deviations of the stabilization angle and of the gyrocamera's rotation angle are

$$\alpha^* = 0.81 \cdot 10^{-3} \approx 2.8', \qquad \beta^* = 0.324 \approx 18.5^\circ$$

Let us mention that if the values of the parameters do not differ greatly from those used in our example we can use the following approximate formulas for standard deviations of the stabilization angle and of the gyrocamera's rotation angle:

$$\alpha^* \approx \frac{\nu\theta^* r n^2}{g} \sqrt{\frac{2\mu AB}{H(H+N)(\zeta_1+\zeta_2)}}, \quad \beta^* \approx \frac{\nu\theta^* A}{H} \sqrt{\left(\frac{rn^2}{g}\right)^2 \frac{2\mu+\zeta_1+\zeta_2}{\zeta_1+\zeta_2} + \frac{\zeta_1^2}{\nu^2}} \qquad (2.21)$$

In our example Formula (2.21) gives

$$\alpha^* \approx 0.82 \cdot 10^{-3}, \qquad \beta^* \approx 0.328$$

which agrees very well with the values obtained from the exact formula. The approximate formulas (2.21) enable us to estimate the influence of each parameter separately.

Comparing Formula (2.21) with (1.22) we notice that as far as the accuracy of stabilization is concerned, the active gyro-stabilizers have considerable advantage over the passive ones.

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